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Hotelling's product differentiation: an infinite-dimensional linear programming approach

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**Economics and Politics Working Paper 72/2017
July 19th, 2017**

**Economics and Politics Research Group
Working Paper Series**

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November, 2016

Abstract

We show that Hotelling's (1929) linear city model of product horizontal differentiation can be rewritten as an infinite-dimensional linear programming problem as in Kantorovich (1942). This reformulation of the Hotelling's model allows us to characterize its optimal allocation through duality theory and to characterize total costs in terms of the marginal contributions of the agents, that is, their shadow-prices. We analyse an industry with uniformly distributed firms and an industry where firms are concentrated and show how the total cost in each case is related to the Hirschmann-Herfindahl index.

Keywords: mass transportation problem, shadow-prices.

JEL classification: C61, D61, L11, L15.

1 Introduction

In this paper I offer a whole new framework to study horizontal product differentiation in the Hotelling's linear city model. I show that Hotelling's model can be reformulated as an infinite-dimensional linear programming problem as in Kantorovich (1942). In particular, the problem of assigning consumers to firms consists in finding a bivariate probability measure that minimizes total cost and whose marginal distributions equal the distributions of consumers and of firms in the industry. Such reformulation allows us to characterize the optimal allocation through duality theory and, in addition, to characterize it in terms of shadow-prices. I analyze an industry with uniformly distributed firms,

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just as in the standard Hotelling linear city model, and an industry where firms are concentrated. In each case, I show how the total cost is related to the Hirschmann-Herfindahl concentration index. In addition, the Radon-Nikodym derivatives of the optimal bivariate measure is shown to be intimately related to the shadow-prices of consumers and firms. Though there are many applications of Hotelling's model to a variety of situations [see, for instance, Graitson (1982) and Anderson, Palma & Thisse (1992) for the two most known surveys], all are based more or less on the same framework and no one, to my knowledge, has ever reframed Hotelling's framework in the way I propose. The infinite-dimensional linear programming framework has been used in general equilibrium theory and other allocational problems [see, for instance, Gretskey, Ostroy, and Zame (1992, 1999, and 2002), Makowsky & Ostroy (2000) and Peñaloza (2009)], but not in the horizontal product differentiation literature.

Here I deal with Hotelling's model of horizontal product differentiation [Hotelling (1929)]. There is horizontal differentiation when consumers' preferences differ over the characteristics of a certain good. A consumer might consider a specific brand characteristic as ideal, but only certain characteristics are available from the suppliers, so the consumer will have to buy from the supplier whose product's characteristic is as close as possible to what she wants. I assume that characteristics are represented scalarly on the unit interval. For example, the unit interval could represent the degree of sweetness of a chocolate bar. The location of a consumer on the unit interval refers to the degree of sweetness she wants. Consumers are heterogeneous in their tastes in the sense that the whole unit interval coincides with the set of consumers. Similarly, firms are located on a finite set of points. The distance between a consumer and a firm is interpreted as the consumer's disutility from buying a chocolate with less than ideal sweetness. The disutility is equivalent to a transportation cost [Shy (1995), p. 145]. The whole story consists in finding the optimal way of assigning consumers to firms.

Since I want to focus on this novel framework, on shadow-prices and their role in the allocational problem, I simplify matters by assuming that firms simply offer the products without any other action. In other words, only consumers decide to move to the nearest firm and the cost of production does not enter into the problem of minimum total cost, that is, I focus on the issue of transportation. Indeed, my goal is to show that this approach provides the same standard solutions but through different ways. Though this is clearly a simplification on the one hand, this method, on the other, leaves the door ajar to a myriad of possible generalizations that cannot be easily contemplated in the standard approach. For instance, the cost of transportation can take many forms. I use the usual Euclidean metric, but it would be just a simple exercise to use a quadratic distance function or any other convex cost function. Consumers can be distributed according to any nonatomic

measure, not just the Lebesgue measure as in the standard approach. The framework here proposed can also incorporate models with multidimensional characteristics and Salop's circular city model [Salop (1979)]. In this case, we just have to redefine the integral according to the domain of the cost function, that is, the metric used to measure disutility.

Section 2 presents the framework. I first consider the case of uniformly distributed firms on the unit interval. Then I analyze the case of concentrated industries. In both cases the optimal total cost is written as an affine transformation of the Hirschmann-Herfindahl index. In section 3 I show that the Hirschmann-Herfindahl index bounds the total cost and that the lowest total cost is achieved under uniform distribution of firms. This relates to the principle of differentiation [Tirole (1993), p. 278]. In section 4 we present some variations of the standard linear city model and show how they all fit into our framework. In particular, we show how the infinite-dimensional linear programming approach can cope with vertical differentiation, multidimensional characteristics, Bertrand games and non-uniform distribution of consumers. Section 5 concludes the paper.

2 Hotelling via Kantorovich

I consider a very simple version of the Hotelling's linear city model, in which there is a linear city of length 1, say the interval $[0, 1]$, a continuum of consumers uniformly distributed with density 1 along the interval, a finite number of firms selling a certain homogeneous product and located along the interval. Regarding the technological structure of the industry, all firms are homogeneous. The only source of heterogeneity among firms is their location on the unit interval. Assume that production is costless and firms all charge the same price, so that their only demand is their market share. Since consumers minimize disutility from transportation, the problem is one of horizontal product differentiation. These assumptions simplify matters, since the allocation of consumers to firms reduces to a transportation cost minimization problem. Indeed, since I want to focus on the assignment aspect of the allocational problem, I do not need to deal with any game-theoretic structure. In the game-theoretical structure of the Hotelling's linear city model, firms compete in prices and the final allocation is given by a Bertrand equilibrium. Even if we wanted to deal with it in a game-theoretical way, we could restrict attention to symmetric equilibria in prices of the Bertrand game. This and the assumption that consumers derive a sufficiently high benefit from the unit consumed and that this benefit is constant across consumers imply that benefits are constant with respect to consumers and firms. Since production is costless, then only transportation costs matter. These simplifications have the sole purpose of helping us focus on the role of transportation costs in the final assignment of consumers to firms. The novelty of the approach resides

in the reframing of the Hotelling's problem as mass transportation problem in the Banach space of bivariate probability measures. Finally, two cases are considered: uniform and non-uniform distribution of firms. Since the distribution of market shares varies with the distribution of firms, the non-uniform case refers to the case of industrial concentration.

2.1 the case of uniform distribution

2.1.1 Primitives of the model

There is a continuum of mass 1 of heterogeneous consumers uniformly distributed on the unit interval $X = [0, 1]$. Accordingly, $[0, 1]$ is endowed with its relative Borel- σ -field $\mathcal{F} = \mathcal{B}[0, 1]$ and the Lebesgue measure λ . Each consumer is denoted by $x \in X$.

There are N identical firms equally spread out on the interval $[0, 1]$. The i^{th} firm is denoted by y_i and its location is given by $y_i = \frac{2i-1}{2N}$, for $i = 1, \dots, N$. Let $Y = \{y_1, \dots, y_N\}$ be the set of firms. Endow Y with the σ -field $\mathcal{G} = 2^Y$ and with the discrete uniform distribution ν defined by $\nu = \frac{1}{N} \sum_{i=1}^N \delta_{y_i}$, for all $y \in Y$, where δ_{y_i} is the Dirac measure concentrated on y_i . In other words, $\nu(y_i) = \frac{1}{N}$, for all $i = 1, \dots, N$.

There is only one good. Every firm sells this good, but the good is differentiated by the location of the firm where it is sold. In all respects except location the good is homogeneous. In order to focus on the role of duality only, assume, for the sake of simplicity, that only consumers incur costs. Production is costless.

The heterogeneity of consumers' tastes is described by the Euclidean proximity of their location to each firm's location. In particular, consumer x has utility function $U(x, y) = \varphi(x) - \|x - y\|$. When consumer x has nil transportation cost, that is, when she remains in her original location, then $\varphi(x) = U(x, x)$ can be interpreted as the utility attained at initial location, thus as the utility from not moving. Any infinitesimal movement from x to $y \neq x$ decreases her utility of akinesia. Therefore, $\varphi(x)$ is x 's kinetic margin. In other words, akinesia happens whenever a consumer finds the good with the exact characteristic she wants. The modulaic transportation cost $\|x - y\|$ means that consumer x gets lower disutility if she moves to the nearest firm than to further away. Each consumer then incurs a transportation cost of \$1 per infinitesimal unit of length. Since we are interested in the transportation problem only, assume, for simplicity, that $\varphi(x) = 0$, for any $x \in X$. In other words, consumer x has utility function $U(x, y) = -\|x - y\|$. Instead of dealing with utility functions, we will deal, instead, with the cost function $c(x, y) = \|x - y\|$, which is the transportation cost from location x to location y . The transportation cost is then the disutility from transportation. The interpretation of a location x occupied by a consumer is that she wants to buy a good with characteristic x , but the good is available only with characteristics y_1, \dots, y_N . Therefore, the transportation cost is indeed

a measure of her disutility or loss of surplus.

The characteristic of a good is a scalar variable and varies *pari passu* with the points in the unit interval. Then $y_i < y_{i+1}$ means not only that the characteristic of the good offered by firm y_{i+1} is higher than the characteristic y_i of the good offered by the firm y_i , also the difference of characteristic can be measured scalarly as the Euclidean distance $\|y_{i+1} - y_i\| = y_{i+1} - y_i$. Therefore, for any $\varepsilon > 0$ not too large, consumers $x = y_i - \varepsilon$ and $x' = y_i + \varepsilon$, should they both be allocated to firm y_i , incur the same transportation cost, $\|x - y_i\| = \|x' - y_i\| = \varepsilon$, but the interpretation for each one differs. To illustrate, suppose that the good is gold. The maximum characteristic of an ounce troy is 24 carats, so we can normalize the range of carats to the unit interval, where $x = 1$ means 24 carats. If there are 6 firms, say goldsmiths, $y_5 = \frac{2 \times 5 - 1}{2 \times 6} = 0.75$ means that firm y_5 offers a unit of gold of $0.75 \times 24 = 18$ carats. Consider two jewelers, say $x = 0.725$ and $x' = 0.775$ who produce some specific type of jewel with gold. Hence they each consume one unit of gold of different carat. The first jeweler sells her product to a less exigent market, so she can produce jewel with less pure gold. The second jeweler produces the same jewel with a purer gold. This rationale suits other markets as well, such as precious stones, wine market and other spirits (differentiated by years of storage), coffee, movie theaters within the city limits, etc. If $N = 6$, then the goldsmiths offer units of gold of, respectively, 2, 6, 10, 14, 18 and 22 carats. They correspond, respectively, to the points $y_1 = 0.083$, $y_2 = 0.250$, $y_3 = 0.417$, $y_4 = 0.583$, $y_5 = 0.750$, and $y_6 = 0.917$. Consumers $x = 0.725$ and $x' = 0.775$ want one unit of gold of, respectively, 17.4 carats and 18.6 carats. According to the Hotelling's model, both consumers move to firm y_5 and incur the same transportation cost, 0.6. From the point of view of consumer $x' = 0.775$, her transportation cost is a measure of the characteristic she had to give up because it was cheaper for her to consume one unit of gold of 17 carats than one of 18 carats. It measures her reduction of characteristic requirement. We call such an agent a *katakinetic* consumer. On the other hand, from the point of view of consumer $x = 0.725$, her transportation cost is a measure of the extra cost she incurs in order to use gold of higher characteristic to produce jewels, so it measures the cost of her characteristic increasing, so she is an *anakinetic* consumer.

2.1.2 The infinite-dimensional linear programming problem

Let $\mathbf{M}(X)$ denote the set of all regular Borel probability measures on X . Consider an element $\mu \in \mathbf{M}_+(X \times Y)$. Let \mathbb{P}_X and \mathbb{P}_Y be the projection operators on X and Y , respectively, applied to μ . In other words, $\mathbb{P}_X[\mu](B) = \mu(B \times Y) = \lambda(B)$, $\forall B \in \mathcal{F}$, and $\mathbb{P}_Y[\mu](C) = \mu(X \times C) = \nu(C)$, $\forall C \in \mathcal{G}$, are the marginal probabilities. Clearly, $\nu(C) = \frac{\#C}{N}$, $\forall C \subset Y$. Thus, μ is a bivariate probability measure whose marginal distributions on

the space X of consumers and on the space Y of firms are equal to the distribution of consumers and the distribution of firms, respectively. Since Y is finite, we can write μ as a vector measure in the sense that $\mu = (\mu_1, \dots, \mu_N) \in \mathbf{M}_+(X)^N$, where $\mu_i \in \mathbf{M}_+(X)$, for all $i = 1, \dots, N$.

The allocation of consumers to firms consists in determining the mass of consumers that choose to buy from each firm. This is given by a vector measure $\mu = (\mu_1, \dots, \mu_N) \in \mathbf{M}_+(X)^N$. The probability measure $\mu_i \in \mathbf{M}_+(X)$ determines the distribution of consumers that buy from firm $i = 1, \dots, N$.

The social planner wants to minimize the transportation total cost, that is, the social cost, defined by $SC = \int_X \int_Y \|x - y\| d\mu(x, y)$, subject to the feasibility constraints.¹

Definition: A measure $\mu \in \mathbf{M}_+(X \times Y)$ is *feasible* if $\mathbb{P}_X[\mu] = \lambda$ and $\mathbb{P}_Y[\mu] = \nu$. The set of all feasible measures is denoted by $\Pi(\lambda, \nu)$.

Thus the problem of optimal allocation in the Hotelling's model can be viewed as a mass-transfer problem as formulated by Kantorovich (1942). Therefore we have to solve the infinite-dimensional linear programming problem:

$$\left\{ \begin{array}{ll} \min & \int_X \int_Y \|x - y\| d\mu(x, y) \\ \text{s.t.} & \mathbb{P}_X[\mu] = \lambda \\ & \mathbb{P}_Y[\mu] = \nu \\ & \mu \in \mathbf{M}_+(X \times Y) \end{array} \right.$$

We call the problem above the *Hotelling primal problem (HPP)*. This general formulation is valid for any pair of marginal distributions, provided they have a compact support.

We can give a fairly general definition of a Hotelling industry, one for which ours is a particular case.

Definition: A *Hotelling industry* is an array $\mathbf{H}_N = (\lambda, \nu, c, N)$, where X is a compact metric space, $Y \subset X$ is finite with cardinality N , $c \in \mathbf{C}(X \times Y)$ is a continuous cost function, and $\lambda \in \mathbf{M}(X)$ and $\nu \in \mathbf{M}(Y)$ are probability measures.

Such formulation allows for different distributions of consumers and firms and different cost functions. Of course the parameters may be very general, but for many purposes only

¹If we had considered a Bertrand game, then we could assume that consumer $x \in (y_{i-1}, y_i)$ have utility function $U(x, y_{i-1}) = b(x) - p_{i-1} - (x - y_{i-1})$, for $x \in (y_{i-1}, \frac{y_i + y_{i-1}}{2}]$, and $U(x, y_i) = b(x) - p_i - (y_i - x)$, for $x \in [\frac{y_i + y_{i-1}}{2}, y_i)$, where p_{i-1} is the price charged by firm y_{i-1} , p_i is the price charged by firm y_i , and $b(x)$ is the benefit, to consumer x , from the consumption of one unit of the good. If we assume that $b(x) = b$, $\forall x \in [0, 1]$, and that the equilibrium is symmetric, $p_{i-1} = p_i$, then the benefit is constant and plays no role in the allocational problem.

simple spaces are needed. Here we consider the case $X = [0, 1]$ with Lebesgue measure $\lambda \in \mathbf{M}[0, 1]$, and Y a set of points uniformly distributed on $[0, 1]$. In the next section we consider non-uniform distributions on $\mathbf{M}(Y)$.

The dual problem associated to *HPP*, which we call the *Hotelling dual problem (HDP)*, is:

$$\begin{cases} \max & \int_X r(x) d\lambda(x) + \int_Y s(y) d\nu(y) \\ s.t. & r(x) + s(y) \leq \|x - y\|, \forall (x, y) \in X \times Y \\ & r \in \mathbf{C}(X), s \in \mathbf{C}(Y) \end{cases}$$

where $\mathbf{C}(X)$ is the space of continuous functions on X and r is the shadow-price function of consumers [see Rachev & Rüschendorf (1998) and Villani (2003)]. Similarly, $s \in \mathbf{C}(Y)$ is the shadow-price function of firms. Since in the Hotelling model, Y is finite, the dual space $\mathbf{C}(Y)$ collapses isomorphically to a finite-dimensional space. I will use *HDP* to find the solution to *HPP*.

Definition: Let $\mathbf{H}_N = (\lambda, \nu, c, N)$ be a Hotelling industry. A probability measure $\mu^* \in \mathbf{M}_+(X \times Y)$ that solves *HPP* above² is called an *optimal Hotelling distributional allocation for \mathbf{H}_N* .

I now present a new proof of the optimal allocation in the linear city model. In particular, I prove that a solution exists and provide a characterization of it in terms of its Radon-Nikodym derivatives. Since any feasible probability measure $\mu \in \mathbf{M}_+(X \times Y)$ is bivariate, there are two Radon-Nikodym derivatives, $\frac{d\mu(x, y_i)}{d\lambda}$ and $\frac{d\mu(x, y_i)}{d\nu}$. Consider firm i whose product has characteristic y_i and let dx be an infinitesimal mass of consumers assigned to firm i . The Lebesgue measure of such an infinitesimal mass of consumers is $\lambda(dx)$. Then $c(x, y_i)\mu(dx, y_i)$ is the cost of assigning the consumers in dx to firm i . Call it the *local cost* of dx with respect to y_i . Then the ratio $\frac{\mu(dx, y_i)}{\lambda(dx)} \cong \frac{d\mu(x, y_i)}{d\lambda}$ is, up to a first order approximation, the average local cost of assignment of consumer x to firm i . If $\mu^* \in \mathbf{M}_+(X \times Y)$ is an optimal Hotelling distributional allocation, then $\frac{d\mu^*(x, y_i)}{d\lambda}$ is the relative parcel of the optimal social cost attributed to the assignment of consumer x to firm i . Now fix consumer x and consider a small mass of characteristics dy_i around y_i . Since Y is finite and y_i is an atom, we have that $\nu(dy_i)$ equals the market share of firm i . Then $\frac{\mu(x, dy_i)}{\nu(dy_i)} \cong \frac{d\mu^*(x, y_i)}{d\nu}$ measures the relative impact on the local cost of assignment between x and y_i of a small change of characteristic.

²Notice that the feasible set is never empty, since it contains at least the product-measure $\mu \otimes \nu$. Moreover, the feasible set is weakly-star compact. Since the cost function (the Euclidean distance in this case, $c(x, y) = \|x - y\|$) is continuous, we only need the mild requirement that the map $\mu \mapsto \int_X \int_Y c(x, y) d\mu(x, y)$ be continuous in the weak-star topology for the problem to admit a solution.

Theorem 1: Consider a Hotelling industry $\mathbf{H}_N = (\lambda, \nu, c, N)$. Let $[0, 1] = \cup_{i=1}^N B_i$, where $B_i = [\frac{i-1}{N}, \frac{i}{N})$, for $i = 1, \dots, N-1$, and $B_N = [\frac{N-1}{N}, 1]$, be a partition of the unit interval, and let s_i^* be the shadow-price of firm $i = 1, \dots, N$, and $r^*(x)$ the shadow-price of consumer $x \in [0, 1]$. Then:

(a) Firms have zero shadow-prices, that is, $s_1^* = \dots = s_N^* = 0$, and consumer x has shadow-price $r^*(x) = \sum_{i=1}^N \|x - y_i\| \mathbf{1}_{B_i}(x)$, $\forall x \in [0, 1]$, so that consumers fully bear their social marginal cost of transportation.

(b) The optimal Hotelling distributional allocation is given by the probability measure $\mu^* \in \mathbf{M}_+(X \times Y)$ for which $\frac{d\mu^*(x, y_i)}{d\lambda} = \sum_{k=1}^N \mathbf{1}_{B_k}(x)$, for each $i = 1, \dots, N$, and $\frac{d\mu^*(x, y_i)}{d\nu} = N\mu^*(x, y_i)$, for almost all $x \in [0, 1]$, where $\frac{d\mu^*(x, y_i)}{d\lambda}$ is the Radon-Nikodym derivative of $\mu^*(\cdot, y_i)$ with respect to the Lebesgue measure λ , $\frac{d\mu^*(\cdot, y_i)}{d\nu}$ is the Radon-Nikodym derivative of $\mu^*(\cdot, y_i)$ with respect to the measure ν . The solution is unique up to a Lebesgue-set of measure zero.

(c) The minimum social cost is equal to $SC_{\min} = \frac{1}{4N}$.

Proof: (a) Clearly the HPP can be rewritten as:

$$\begin{cases} \min & \sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu(x, y_i) \\ \text{s.t.} & \mathbb{P}_X[\mu] = \lambda \\ & \mathbb{P}_Y[\mu] = \nu \\ & \mu \in \mathbf{M}_+(X \times Y) \end{cases}$$

The dual problem is given by:

$$\begin{cases} \max & \int_0^1 r(x) d\lambda(x) + \frac{1}{N} \sum_{i=1}^N s_i \\ \text{s.t.} & r(x) + s_i \leq \|x - y_i\|, \forall x \in [0, 1], \forall i = 1, \dots, N \\ & r \in \mathbf{C}[0, 1] \end{cases}$$

where $r(x)$ is the shadow-price of consumer x and s_i is the shadow-price of firm i . Clearly, $s_i = \inf\{\|x - y_i\| - r(x) : x \in [0, 1]\}$. Define $\psi_i(x) = \|x - y_i\| - r(x)$. Even though $r(\cdot)$ is restricted to be continuous, it is easy to see that we can assume it is differentiable almost everywhere. In fact, we will check ahead that the solution $r^*(\cdot)$ is not differentiable only at the points occupied by the firms, which comprise a finite set. Let $x \in [\frac{i-1}{N}, \frac{i}{N})$. Recall that $y_i = \frac{2i-1}{2N}$ is the middle point of this subinterval. If $x > y_i$, then $\psi_i(x) = x - y_i - r(x)$. In this case, $\frac{\partial \psi_i}{\partial x} = 0$ implies $1 - r'(x) = 0$, i.e., $r'(x) = 1$, so that $r(x) = x + A_i$, where A_i is a constant of integration. If $x < y_i$, then $\psi_i(x) = y_i - x - r(x)$. In this case, $\frac{\partial \psi_i}{\partial x} = 0$ implies $r'(x) = -1$, hence $r(x) = -x + C_i$, where C_i is also a constant of integration. Notice that $r'(x)$ is discontinuous only at the points y_1, \dots, y_N . Otherwise it is a constant, hence the function r is piecewise continuous³ and nondifferentiable only

³It is actually a 1-Lipschitz-continuous function.

at the points at where firms are located. Since r is continuous, we have $r(y_i+) = r(y_i-)$, that is, $\lim_{x \uparrow y_i} r(x) = \lim_{x \downarrow y_i} r(x)$. Therefore, $y_i + A_i = -y_i + C_i$. Then $y_i = \frac{C_i - A_i}{2}$, that is, $2y_i = C_i - A_i$. But the individual located at y_i has obviously nil shadow-price, that is, $r(y_i) = 0$. Hence, $y_i + A_i = 0$ and $-y_i + C_i = 0$. Therefore, $A_i = -y_i$ and $C_i = y_i$. In particular, these constants of integration clearly satisfy the condition $\frac{C_i - A_i}{2} = y_i$. Hence $r^*(x) = \sum_{i=1}^N \|x - y_i\| \mathbf{1}_{B_i}(x)$, $\forall x \in [0, 1]$, where $B_i = [\frac{i-1}{N}, \frac{i}{N})$, for $i = 1, \dots, N-1$, and $B_N = [\frac{N-1}{N}, 1]$. This implies that the solution set for $\inf\{\|x - y_i\| - r(x) : x \in [0, 1]\}$ is the closure of B_i , that is, $\overline{B}_i = [\frac{i-1}{N}, \frac{i}{N}]$. It follows that $s_i^* = 0$, for all $i = 1, \dots, N$. In other words, firms have zero shadow-prices.

(b) Since $X = [0, 1]$, in particular it is a Polish space. The cost function $c : X \times Y \rightarrow \mathbb{R}_+ \cup \{\infty\}$ defined by $c(x, y) = \|x - y\|$ is lower semicontinuous. Since r is continuous, then $r \in \mathbf{L}^1(\lambda)$ (i.e., it is Lebesgue-integrable on X) and $s \in \mathbb{R}^n$, then, by the Kantorovich duality theorem, there is no duality gap⁴ [Kantorovich (1942) and Villani (2003)]. Since there is no duality gap, primal and dual values are equal, that is:

$$\sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu^*(x, y_i) = \int_0^1 r^*(x) d\lambda(x)$$

Notice that $\int_0^1 r^*(x) d\lambda(x) = \sum_{i=1}^N \int_0^1 \|x - y_i\| \mathbf{1}_{B_i}(x) d\lambda(x)$. Therefore:

$$\sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu^*(x, y_i) = \sum_{i=1}^N \int_0^1 \|x - y_i\| \mathbf{1}_{B_i}(x) d\lambda(x)$$

Consider the duality bracket $\langle \mathbf{C}(X \times Y), \mathbf{M}(X \times Y) \rangle$, where $\mathbf{M}(X \times Y)$ is the topological dual space of $\mathbf{C}(X \times Y)$, and let $\langle f, \mu \rangle = \int_X \int_Y f(x, y) d\mu(x, y)$ be the evaluation function. For each $y \in Y$, let $B_i \in \mathcal{F}$ be the only B_i for which $y \in B_i$. Call it $\tau(y)$. Clearly τ is a well-defined correspondence. Define $v(S \times T) = \sum_{y \in T} \lambda(S \cap \tau(y))$, for all measurable rectangles $S \times T \in \mathcal{F} \otimes \mathcal{G}$.⁵ Thus $v(S \times T) = \sum_{y_i \in T} \lambda(S \cap B_i)$. But $\sum_{y_i \in T} \lambda(S \cap B_i) = \int_{S \cap B_i} \sum_{y_i \in T} d\lambda(x)$, so that $v(S \times T) = \int_{S \cap B_i} \sum_{y_i \in T} d\lambda(x)$. Therefore, $v(S \times T) = N \int_S \int_T \mathbf{1}_{B_i}(x) d\lambda(x) d\nu(y)$. Recall that $d\nu(y) = \frac{1}{N}$, $\forall y \in Y$, so $v(S \times \{y_i\}) = \int_S \mathbf{1}_{B_i}(x) d\lambda(x)$. Hence $dv(x, y_i) = \mathbf{1}_{B_i}(x) d\lambda(x)$. In our case, $\langle c, \mu^* \rangle = \sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu^*(x, y_i)$. Since $\langle c, \mu^* \rangle = \langle c, v \rangle$, we have $\langle c, \mu^* - v \rangle = 0$, independently of the cost function $c \in \mathbf{C}(X \times Y)$, which implies $\mu^* = v$.

We will show that μ^* is feasible.

We have that $d\mu^*(x, Y) = \sum_{i=1}^N \mathbf{1}_{B_i}(x) d\lambda(x)$, from which we get, $\forall B \in \mathcal{F}$, $\mu^*(B, Y) = \int_B \sum_{i=1}^N \mathbf{1}_{B_i}(x) d\lambda(x)$, that is, $\mu^*(B, Y) = \sum_{i=1}^N \int_{B \cap B_i} d\lambda(x)$. Since $\int_{B \cap B_i} d\lambda(x) = \lambda(B \cap B_i)$, we have that $\mu^*(B, Y) = \sum_{i=1}^N \lambda(B \cap B_i)$. Given that $[0, 1] = \cup_{i=1}^N B_i$ is a disjoint

⁴The finiteness of Y poses no problem to this result.

⁵Notice that $\sum_{y \in T} \lambda(S \cap \tau(y)) = \sum_{y_i \in T} v(S \times \{y_i\})$, so $v(S \times T) = \sum_{y_i \in T} v(S \times \{y_i\})$.

partition, it follows that $\sum_{i=1}^N \lambda(B \cap B_i) = \lambda(B \cap (\cup_{i=1}^N B_i))$, hence $\mu^*(B, Y) = \lambda(B)$. Moreover, from $d\mu^*(x, y_i) = \mathbf{1}_{B_i}(x)d\lambda(x)$, we get $\mu^*([0, 1], y_i) = \int_0^1 \mathbf{1}_{B_i}(x)d\lambda(x)$, hence $\mu^*([0, 1], y_i) = \int_{B_i} d\lambda(x)$, which implies that $\mu^*([0, 1], y_i) = \lambda(B_i)$. Since $\lambda(B_i) = \frac{1}{N}$ is the market share of firm i and this is given by $\nu(y_i) = \frac{1}{N}$, we have that $\mu^*([0, 1], y_i) = \nu(y_i)$, that is, $\mathbb{P}_X[\mu] = \lambda$ and $\mathbb{P}_Y[\mu] = \nu$.

Finally, note that μ^* is a probability measure. Indeed, we have:

$$\mu^*([0, 1], Y) = \sum_{i=1}^N \int_0^1 d\mu^*(x, y_i)$$

But $\sum_{i=1}^N \int_0^1 d\mu^*(x, y_i) = \int_0^1 N \mathbf{1}_{B_i}(x)d\lambda(x)$, which equals $N\lambda(B_i)$. Since $\lambda(B_i) = \frac{1}{N}$, for all $i = 1, \dots, N$, we finally get $\mu^*([0, 1], Y) = 1$, that is, $\mu^* \in \mathbf{M}_+(X \times Y)$.

Therefore, μ^* is feasible.

Let us now characterize the solution in terms of its Radon-Nikodym derivatives with respect to the distribution of consumers and firms. It was already shown that $\frac{d\mu^*(x, y_i)}{d\lambda} = \mathbf{1}_{B_i}(x)$. In other words, the Radon-Nikodym derivative of the optimal Hotelling-allocation with respect to the distribution of consumers prescribes that consumers within the interval $B_i = [\frac{i-1}{N}, \frac{i}{N})$ move to the firm $y_i = \frac{2i-1}{2N}$, which is the center point of that interval. The Radon-Nikodym derivative $\frac{d\mu^*(x, y_i)}{d\nu}$ of the optimal allocation with respect to the distribution of firms is trivially given by $\frac{d\mu^*(x, y_i)}{d\nu} = \frac{\mu^*(x, y_i)}{\nu(y_i)}$, hence $\frac{d\mu^*(x, y_i)}{d\nu} = N\mu^*(x, y_i)$, since ν is discrete, which proves the result.

(c) Finally, the total cost corresponding to the optimal Hotelling distributional allocation is $SC_{\min} = \sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu^*(x, y_i)$. Since the solution is characterized by $d\mu^*(x, y_i) = \mathbf{1}_{B_i}(x)d\lambda(x)$, we have that $SC_{\min} = \sum_{i=1}^N \int_0^1 \|x - y_i\| \mathbf{1}_{B_i}(x)d\lambda(x)$, that is, $SC_{\min} = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \|x - y_i\| d\lambda(x)$. Given that $y_i = \frac{2i-1}{2N}$ Therefore:

$$\begin{aligned} SC_{\min} &= \sum_{i=1}^N \left[\int_{\frac{i-1}{N}}^{\frac{2i-1}{2N}} \left(\frac{2i-1}{2N} - x \right) d\lambda(x) + \int_{\frac{2i-1}{2N}}^{\frac{i}{N}} \left(x - \frac{2i-1}{2N} \right) d\lambda(x) \right] \\ &= \frac{1}{4N} \end{aligned}$$

hence $SC_{\min} = \frac{1}{4N}$. This completes the proof. ■

Scholium: In the proof above we obtained μ^* through the dual linear programming problem and the no-gap theorem, so μ^* is indeed the optimal solution for the primal linear programming problem, *HPP*. We can also give a more heuristic argument. Let μ be any feasible allocation other than μ^* . Since Y is finite, any such deviation should be of the form $d\mu(x, y_i) = \mathbf{1}_{D_i}(x)d\lambda(x)$, where $[0, 1] = \cup_{i=1}^N D_i$ is a nontrivial partition of $[0, 1]$ and, for some $i_o \in \{1, \dots, N\}$ and some sufficiently small $\varepsilon > 0$, $\lambda(D_{i_o} \cap B_{i_o}^c) \geq \varepsilon$, where

$B_{i_o}^c = [0, 1] \setminus B_{i_o}$, and where we assume, without loss of generality, that, apart from the sets indexed by i_o and $i_o - 1$, everything else remains the same. Take $\varepsilon < \frac{1}{2N}$. Assume then, for simplicity, that only those two adjacent segments are altered, in the sense that consumers who otherwise remain faithful to one firm, move instead to the adjacent firm. In terms of the shadow-price function above, this means that the market segment appropriated by the firm y_{i_o} gets an increasing of share (say, to the left) by the amount ε and the adjacent one to the left, y_{i_o-1} , gets a decreasing by the same amount. Therefore, the area below the shadow-price function corresponding to the new market share of firm y_{i_o} increases by the amount $\rho^+ = \frac{1}{4} \left(\frac{1}{N} + \varepsilon \right)^2 - \frac{1}{4N^2}$. This is the incremental total cost due to the switch. On the other hand, the area below the shadow-price function corresponding to the new market share of firm y_{i_o-1} decreases by the amount $\rho^- = \frac{1}{4N^2} - \frac{1}{4} \left(\frac{1}{N} - \varepsilon \right)^2$. This is a saving of total cost. However, after simple algebraic calculation, we easily verify that $\rho^+ - \rho^- = \frac{\varepsilon^2}{2} > 0$, hence the incremental total cost is greater than the incremental saving. This means that any small perturbation of μ^* increases the total cost. If the perturbation is greater than $\frac{1}{2N}$, this will affect the next to the adjacent firm, and so on, so that costs only increases. Therefore, μ^* is indeed optimal. ■

It follows from theorem 1 that the parcel $SC(x)$ of the social cost due to the assignment of consumer $x \in B_i$ to firm y_i is fully born by the consumer, that is, she internalizes, as private cost, her own shadow-price: $SC(x) = r^*(x)$. This is the economic value of consumer x . Consumers positioned at the equidistant point between two consecutive firms have the highest shadow-price, $\frac{1}{2N}$. These are the consumers who are indifferent between two consecutive firms. It is actually straightforward that firms have zero shadow-prices, since all they have to do is being at their locations and wait for the matches to be done. The cost they incur when offering their product was neglected in the social planner's problem, since the focus was put on consumers only and the dual framework. This shows that, should one firm be withdrawn from the economy, consumers would be reallocated to the remaining firms without any change in the social cost of transportation.

2.1.3 The rôle of Radon-Nikodym derivatives

One of the insights we get from our duality approach to the Hotelling's model is that we can obtain this same conclusion from Radon-Nikodym derivatives. Let us first illustrate this with regard to the shadow-prices of firms. Recall that $\frac{d\mu^*(x, y_i)}{d\nu} = N\mu^*(x, y_i)$. Consumer x is assigned to a good of characteristic y_i . Consider an infinitesimal neighborhood dy_i of characteristics around y_i . Then $\nu(dy_i) = \frac{1}{N}$, since y_i is an atom. Given that $\frac{d\mu^*(x, y_i)}{d\nu} \cong \frac{\mu^*(x, dy_i)}{\nu(dy_i)}$, we therefore have $\mu^*(x, dy_i) = \mu^*(x, y_i)$. In other words, a sufficiently small change of characteristic does not have any effect on the social weight $\mu^*(x, y_i)$ at-

tributed by society to the assignment of consumer x to firm i . Thus, to a first order approximation, firm i has no allocational impact, hence no effect on the optimal social cost. This means it has zero shadow-price.

Consider now $\frac{d\mu^*(x, y_i)}{d\lambda} = \mathbf{1}_{B_i}(x)$. Then $\mu^*(dx, y_i) = \mathbf{1}_{B_i}(x)\lambda(dx)$. The characteristic y_i is fixed and consumer $x \in B_i$ is assigned to firm i . Any consumer in a small neighborhood dx of x values characteristic y_i in a way very similar to the value x attributes to y_i . If consumer x 's valuation of characteristic y_i varies by dx , then, should she still belong to B_i , the social weight $\mu^*(x, y_i)$ of the local cost $c(x, y_i)$ varies by $\lambda(dx)$, that is, exactly the variation of the distance between the characteristic x that the consumer wants and the closest characteristic y_i available to her. In other words, the shadow-price of consumers' valuation of characteristics is given by the distance function, that is, the disutility of transportation. Since this reasoning applies to every consumer and every firm, $r^*(x) = \sum_{i=1}^N \|x - y_i\| \mathbf{1}_{B_i}(x)$.

Therefore, Radon-Nikodym derivatives of the optimal Hotelling distributional allocation and shadow-prices of consumers and firms are intimately related in the sense that mere interpretations of the Radon-Nikodym derivatives provide much information about the shadow-prices of agents obtained through our duality approach.

2.2 the case of concentrated industries

In this section I consider the case in which the qualities of the product offered are not uniformly distributed. From the very structure of the Hotelling's linear city model, market shares will not be equal, so the industry will be concentrated.

Again, there are N identical firms. They are unequally spread out on the interval $[0, 1]$. The location of the i^{th} firm is denoted by $y_i \in [0, 1]$. Assume that $0 < y_1 < \dots < y_N < 1$. Let $Y = \{y_1, \dots, y_N\}$ be the set of firms. Endow Y with the σ -field $\mathcal{G} = 2^Y$ and with the discrete distribution ν defined by $\nu = \sum_{i=1}^N \beta_i \delta_{y_i}$, where δ_{y_i} is the Dirac measure concentrated on y_i , $0 < \beta_i < 1$, for all $i = 1, \dots, N$, and $\sum_{i=1}^N \beta_i = 1$. In other words, $\nu(y_i) = \beta_i$. Everything else in the model is the same as in the previous section. Since it is a location model in which consumers are matched with the closest firm, it follows that the mass β_i of firm y_i must be the mass of consumers with whom that firm will certainly match. The framework of the Hotelling's model makes the firm capture consumers located within the segment whose extremes are the middle points between the firm itself and its adjacent firms. Then firm y_i will capture consumers located within the interval $[\frac{y_{i-1} + y_i}{2}, \frac{y_i + y_{i+1}}{2})$. The ones located on the edges are indifferent, so we arbitrarily assigne the consumer on the left-hand side extreme to firm y_i . The consumer at point $x = 1$ is assigned arbitrarily to the firm y_N . Therefore, we necessarily have to assume that β_i equals the Lebesgue

measure of this interval, that is, $\beta_i = \frac{y_{i+1}-y_{i-1}}{2}$. Thus the weight of firm y_i must be its own market share. Recall that the Hirschmann-Herfindahl index is $\mathcal{H}_N = \sum_{i=1}^N \beta_i^2$, the is, the sum total of squared market shares.⁶

Notice, in addition, that, from the parcel of consumers assigned to firm y_i , the set $[\frac{y_{i-1}+y_i}{2}, y_i)$ is formed by consumers who move rightwards, that is, anakinetic consumers, whereas the set $[y_i, \frac{y_i+y_{i+1}}{2})$ is formed by consumers who move leftwards, the katakinetic consumers. Their respective masses will be denoted by π_i^- and π_i^+ , that is, $\pi_i^- = \frac{y_i-y_{i-1}}{2}$ and $\pi_i^+ = \frac{y_{i+1}-y_i}{2}$. This implies that $\beta_i = \pi_i^+ + \pi_i^-$. Let $\gamma_i = \sqrt{\pi_i^+ \times \pi_i^-}$ be the geometric mean of these parcels and let $\gamma = (\gamma_1, \dots, \gamma_N)$ be the vector of these means. Finally, given the vectors $\pi^- = (\pi_1^-, \dots, \pi_N^-)$ and $\pi^+ = (\pi_1^+, \dots, \pi_N^+)$, consider $\sigma^- = \|\pi^-\|^2$ and $\sigma^+ = \|\pi^+\|^2$. The number σ^- can be interpreted as a measure of concentration of anakinetic consumers, up to a multiplicative constant. Likewise, σ^+ is a measure of concentration of katakinetic consumers. Indeed, notice that $\sum_{i=1}^N \pi_i^-$ is a telescopic sum, so $\sum_{i=1}^N \pi_i^- = y_N$. Similarly, $\sum_{i=1}^N \pi_i^+ = 1 - y_1$. Thus $\alpha = \sigma^-/y_N^2$ is the Hirschmann-Herfindahl index of anakinetic consumers, so it is a measure of *anakinesis*. The higher this measure, the more spread out is the set of anakinetic agents, that is, consumers who have to consume the good with a higher characteristic. In the same vein, $\kappa = \sigma^+/(1 - y_1)^2$ is the Hirschmann-Herfindahl index of katakinetic consumers, so it is a measure of *katakinesis*. The higher this measure, the more spread out is the set of katakinetic agents, that is, consumers who have to consume the good with a lower characteristic.

For ease of notation, define $y_o = 0$ and $y_{N+1} = 1$, where $\nu(y_o) = \nu(y_{N+1}) = 0$.

The problem of optimal allocation in the case of concentrated industries is given by the same infinite-dimensional linear programming problem *HHP* in its general formulation.

Theorem 2: Consider a concentrated Hotelling industry $\mathbf{H}_N = (\lambda, \nu, c, N)$. Then:

(a) The firms have zero shadow-prices, that is, $s_1^* = \dots = s_N^* = 0$, and consumer x has shadow-price $r^*(x) = \sum_{i=1}^N \|x - y_i\| \mathbf{1}_{B_i}(x)$, $\forall x \in [0, 1]$, so that consumers fully bear their social marginal cost of transportation.

(b) The optimal Hotelling distributional allocation of the Hotelling assignment problem is given by the measure $\mu^* \in \mathbf{M}_+(X \times Y)$ for which $\frac{d\mu^*(x, y_i)}{d\lambda} = \mathbf{1}_{B_i}(x)$, for each fixed $i = 1, \dots, N$, where $B_i = [\frac{y_{i-1}+y_i}{2}, \frac{y_i+y_{i+1}}{2})$, for $i = 1, \dots, N-1$, $B_0 = [0, \frac{y_1}{2})$, and $B_N = [\frac{y_{N-1}+y_N}{2}, 1]$, and for which $\frac{d\mu^*(x, y_i)}{d\nu} = \frac{1}{\beta_i} \mu^*(x, y_i)$, for almost all $x \in [0, 1]$, where $\frac{d\mu^*(x, y_i)}{d\lambda}$ is the Radon-Nikodym derivative of $\mu^*(\cdot, y_i)$ with respect to the Lebesgue measure λ , $\frac{d\mu^*(\cdot, y_i)}{d\nu}$ is the Radon-Nikodym derivative of $\mu^*(\cdot, y_i)$ with respect to the measure ν . The solution is unique up to a Lebesgue-set of measure zero.

⁶However, we have to keep in mind that this was so only because the measure on the space of consumers is given by the Lebesgue measure, which is generated by the Euclidian distance.

(c) The minimum social cost is equal to $SC_{\min} = \frac{\sigma^- + \sigma^+}{2}$ or, alternatively, $SC_{\min} = \frac{\mathcal{H}_N}{2} - \|\gamma\|^2$.

Proof: (a) Clearly problem *HPP* can be rewritten as:

$$\begin{cases} \min & \sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu(x, y_i) \\ \text{s.t.} & \mathbb{P}_X[\mu] = \lambda \\ & \mathbb{P}_Y[\mu] = \nu \\ & \mu \in \mathbf{M}_+(X \times Y) \end{cases}$$

The dual problem is given by:

$$\begin{cases} \max & \int_0^1 r(x) d\lambda(x) + \sum_{i=1}^N \beta_i s_i \\ \text{s.t.} & r(x) + s_i \leq \|x - y_i\|, \forall x \in [0, 1], \forall i = 1, \dots, N \\ & r \in \mathbf{C}[0, 1] \end{cases}$$

where $r(x)$ is the shadow-price of consumer x and s_i is the shadow-price of firm i . We have that $s_i = \inf\{\|x - y_i\| - r(x) : x \in [0, 1]\}$. Define $\psi_i(x) = \|x - y_i\| - r(x)$. If $x > y_i$, then, in a sufficiently small neighborhood of y_i , we have $\psi_i(x) = x - y_i - r(x)$. In this case, $\frac{\partial \psi_i}{\partial x} = 0$ implies $1 - r'(x) = 0$, so that $r'(x) = 1$, hence $r(x) = x + A_i$, where A_i is a constant of integration. If $x < y_i$, then $\psi_i(x) = -x + y_i - r(x)$. Thus, $\frac{\partial \psi_i}{\partial x} = 0$ implies $-1 - r'(x) = 0$, so that $r'(x) = -1$, that is, $r(x) = -x + C_i$. Since r is continuous, we have $r(y_i+) = r(y_i-)$, that is, $\lim_{x \uparrow y_i} r(x) = \lim_{x \downarrow y_i} r(x)$. Therefore, $y_i + A_i = -y_i + C_i$. Then $y_i = \frac{C_i - A_i}{2}$, that is, $2y_i = C_i - A_i$. The constants of integration must satisfy this equation. But the individual located at y_i has obviously nil shadow-price, that is, $r(y_i) = 0$. Hence, $y_i + A_i = 0$ and $-y_i + C_i = 0$. Therefore, $A_i = -y_i$ and $C_i = y_i$. In particular, these constants of integration clearly satisfy the condition $\frac{C_i - A_i}{2} = y_i$. But $y_i + A_i = 0$ and $-y_i + C_i = 0$ also imply $A_i = -C_i$. Then we have the system of equations given by $A_i - C_i = -2y_i$ and $A_i + C_i = 0$, whose unique solution is $A_i = -y_i$ and $C_i = y_i$. Consider the individual $x \in (y_i, y_{i+1})$ who is indifferent between firms y_i and y_{i+1} . Then, since her shadow-price is unique, $x + A_i = -x + C_{i+1}$, hence $x_{i,i+1} = \frac{C_{i+1} - A_i}{2}$. Therefore, $x_{i,i+1} = \frac{y_{i+1} + y_i}{2}$. In other words, the consumer located at the middle point between firms y_i and y_{i+1} is indeed indifferent between them. The solution is then $r(x) = x - y_i$, if $x \geq y_i$, and $r(x) = -(x - y_i)$, if $x < y_i$, all this for $x \in [\frac{y_{i-1} + y_i}{2}, \frac{y_i + y_{i+1}}{2})$, that is, $r(x) = \|x - y_i\|$, for $x \in [\frac{y_{i-1} + y_i}{2}, \frac{y_i + y_{i+1}}{2})$. In other words, $r^*(x) = \sum_{i=1}^N \|x - y_i\| \mathbf{1}_{B_i}(x)$, where $B_i = [\frac{y_{i-1} + y_i}{2}, \frac{y_i + y_{i+1}}{2})$, for $i = 1, \dots, N-1$, $B_0 = [0, \frac{y_1}{2})$, and $B_N = [\frac{y_{N-1} + y_N}{2}, 1]$. The function $r^*(x) = \sum_{i=1}^N \|x - y_i\| \mathbf{1}_{B_i}(x)$ is the dual solution that specify the shadow-price of each consumer x . In other words, the shadow-price of consumer $x \in B_i$ is given by her disutility or private cost of transportation from x to the firm $y_i \in B_i$ to which she is

matched. The graph below depicts the case for $N = 4$.

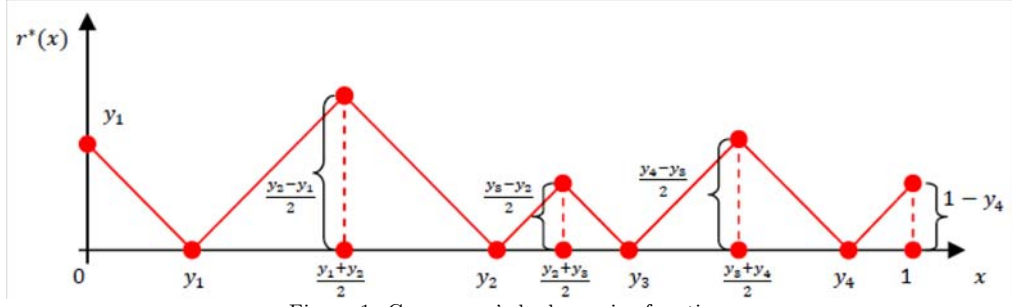


Figure 1. Consumers' shadow-price function.

By assumption, $X = [0, 1]$, so in particular it is a compact and separable metrizable space, hence it is a Polish space. The cost function $c : X \times Y \rightarrow \mathbb{R}_+ \cup \{\infty\}$ defined by $c(x, y) = \|x - y\|$ is lower semicontinuous. Since r is continuous, then $r \in \mathbf{L}^1(\lambda)$ (i.e., it is Lebesgue-integrable on X) and $s \in \mathbb{R}^n$, then, by the Kantorovich duality theorem, there is no duality gap [Kantorovich (1942) and Villani (2003)]. Since there is no duality gap:

$$\sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu^*(x, y_i) = \int_0^1 r^*(x) d\lambda(x) + \sum_{i=1}^N \beta_i s_i$$

We have that $s_i = \inf\{\|x - y_i\| - r(x) : x \in [0, 1]\}$. A quick inspection at the graphs of $\|x - y_i\|, \forall x \in [0, 1]$, and r^* shows us that the infimum is attained at any $x \in [\frac{y_{i-1}+y_i}{2}, \frac{y_i+y_{i+1}}{2}]$, so that $s_i^* = 0$. Therefore, consumers fully bear the total transportation cost.

(b) Given the shadow-prices above and by the no-gap theorem, we have that:

$$\begin{aligned} \sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu^*(x, y_i) &= \int_0^1 r^*(x) d\lambda(x) \\ &= \int_0^1 \sum_{i=1}^N \|x - y_i\| \mathbf{1}_{B_i}(x) d\lambda(x) \\ &= \sum_{i=1}^N \int_0^1 \|x - y_i\| \mathbf{1}_{B_i}(x) d\lambda(x) \end{aligned}$$

The optimal Hotelling-allocation is then characterized, in terms of one of its Radon-Nikodym derivatives, by $d\mu^*(x, y_i) = \mathbf{1}_{B_i}(x) d\lambda(x)$.

Let us now show that μ^* is feasible.

We have $d\mu^*(x, Y) = \sum_{i=1}^N \mathbf{1}_{B_i}(x) d\lambda(x)$, from which we get, $\forall B \in \mathcal{F}$, $\mu^*(B, Y) = \int_B \sum_{i=1}^N \mathbf{1}_{B_i}(x) d\lambda(x)$, so that $\mu^*(B, Y) = \sum_{i=1}^N \int_{B \cap B_i} d\lambda(x)$. Since $\sum_{i=1}^N \int_{B \cap B_i} d\lambda(x) = \sum_{i=1}^N \lambda(B \cap B_i)$, we have that $\mu^*(B, Y) = \sum_{i=1}^N \lambda(B \cap B_i)$. Therefore we get $\mu^*(B, Y) = \lambda(B \cap (\cup_{i=1}^N B_i))$, hence $\mu^*(B, Y) = \lambda(B)$, since $\cup_{i=1}^N B_i = [0, 1]$. Moreover, from $d\mu^*(x, y_i) =$

$\mathbf{1}_{B_i}(x)d\lambda(x)$, we get $\mu^*([0, 1], y_i) = \int_0^1 d\mu^*(x, y_i)$. Since $\int_0^1 d\mu^*(x, y_i) = \int_0^1 \mathbf{1}_{B_i}(x)d\lambda(x)$ and since $\int_0^1 \mathbf{1}_{B_i}(x)d\lambda(x) = \int_{B_i} d\lambda(x)$, we get $\mu^*([0, 1], y_i) = \lambda(B_i)$. Notice that $\lambda(B_i) = \frac{y_{i+1} - y_{i-1}}{2}$ is just firm i 's market share, that is, $\lambda(B_i) = \beta_i$, which satisfies $\beta_i = \nu(y_i)$. Therefore, $\mu^*([0, 1], y_i) = \nu(y_i)$.

In other words, $\mathbb{P}_X[\mu] = \lambda$ and $\mathbb{P}_Y[\mu] = \nu$.

Finally, note that μ^* is a probability measure. Indeed, since $\lambda(B_i) = \beta_i$, for all $i = 1, \dots, N$, we have $\mu^*([0, 1], Y) = \sum_{i=1}^N \int_0^1 d\mu^*(x, y_i)$. Since obviously $\int_0^1 d\mu^*(x, y_i) = \mu^*([0, 1], y_i)$, we then have $\mu^*([0, 1], Y) = \sum_{i=1}^N \mu^*([0, 1], y_i)$. Recall that $\mu^*([0, 1], y_i) = \beta_i$. Therefore

$$\begin{aligned} \mu^*([0, 1], Y) &= \sum_{i=1}^N \beta_i \\ &= 1 \end{aligned}$$

that is, $\mu^* \in \mathbf{M}_+(X \times Y)$. Therefore, μ^* is feasible.

We will now characterize the solution in terms of its Radon-Nikodym derivatives with respect to the distribution of consumers and firms. We have already shown that $\frac{d\mu^*(x, y_i)}{d\lambda} = \mathbf{1}_{B_i}(x)$. In other words, the Radon-Nikodym derivative of the optimal Hotelling-allocation with respect to the distribution of consumers prescribe that consumers within the interval $B_i = [\frac{y_{i-1} + y_i}{2}, \frac{y_i + y_{i+1}}{2})$ move to the firm y_i . The Radon-Nikodym derivative $\frac{d\mu^*(x, y_i)}{d\nu}$ of the optimal allocation with respect to the distribution of firms is trivially given by $\frac{d\mu^*(x, y_i)}{d\nu} = \frac{\mu^*(x, y_i)}{\nu(y_i)}$, hence $\frac{d\mu^*(x, y_i)}{d\nu} = \frac{1}{\beta_i} \mu^*(x, y_i)$, since ν is discrete, which proves the result.

(c) It is easy to calculate, by a simple inspection of the graph of the shadow-price function r , the integral $\int_0^1 r^*(x)d\lambda(x)$. Indeed, by adding the areas of the triangles, rather than calculating the integral, we get:

$$\begin{aligned} \int_0^1 r^*(x)d\lambda(x) &= \frac{1}{2}y_1^2 + \frac{(y_2 - y_1)^2}{4} + \dots + \frac{(y_N - y_{N-1})^2}{4} + \frac{1}{2}(1 - y_N)^2 \\ &= \frac{1}{2}y_1^2 + \frac{1}{2}(1 - y_N)^2 + \frac{1}{4} \sum_{i=1}^{N-1} (y_{i+1} - y_i)^2 \end{aligned}$$

Then the social cost is $SC = \frac{1}{2}y_1^2 + \frac{1}{2}(1 - y_N)^2 + \frac{1}{4} \sum_{i=1}^{N-1} (y_{i+1} - y_i)^2$. If we define $y_0 = 0$ and $y_{N+1} = 1$, then $SC = \frac{1}{4} \sum_{i=0}^N (y_{i+1} - y_i)^2 + \frac{y_1^2 + (1 - y_N)^2}{4}$.

We can, however, get an alternative expression for the total cost in terms of the Hirschmann-Herfindahl index. Take, for instance, firm y_2 in the graph 2 below. Its market share, β_2 , is given by the interval between the points $\frac{y_1 + y_2}{2}$ and $\frac{y_2 + y_3}{2}$, hence β_2^2 is equal to the area of the square given by the dashed triangles plus the tilted rectangle. Since the shadow-price function is piecewise linear and the lines always form a 45° angle (slopes are either $+1$ or -1), the horizontally dashed area equals the vertically dashed area, hence

the whole dashed area equals twice the integral $\int_{\frac{y_1+y_2}{2}}^{\frac{y_2+y_3}{2}} r^*(x) d\lambda(x)$. It is straightforward to calculate the area of the tilted rectangle, which is $\frac{1}{2}(y_3 - y_2)(y_2 - y_1)$.

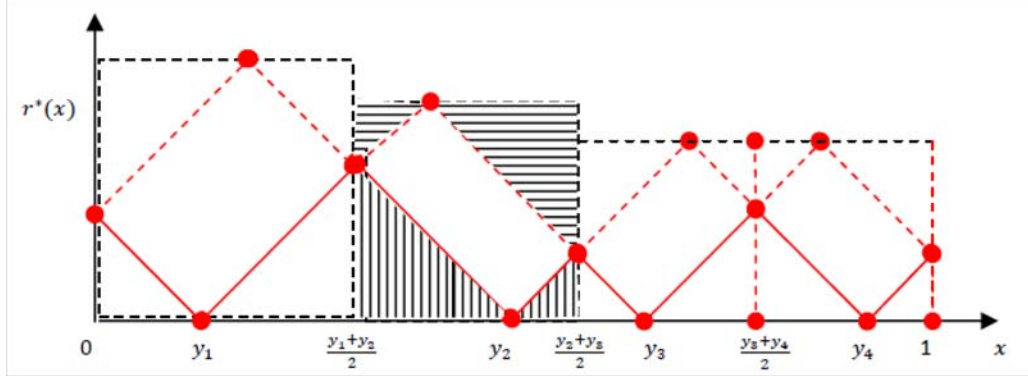


Figure 2. Relation between consumers' shadow-price function and the firms' squared market shares.

Therefore, $\beta_2^2 = 2 \int_{\frac{y_1+y_2}{2}}^{\frac{y_2+y_3}{2}} r^*(x) d\lambda(x) + \frac{1}{2}(y_3 - y_2)(y_2 - y_1)$. This reasoning applies to the general case with N firms. Since we defined $y_o = 0$ and $y_{N+1} = 1$, if we add up all these numbers, we get:

$$\begin{aligned}
\mathcal{H}_N &= \sum_{i=1}^N \beta_i^2 \\
&= 2 \sum_{i=1}^N \int_{\frac{y_{i-1}+y_i}{2}}^{\frac{y_i+y_{i+1}}{2}} r^*(x) d\lambda(x) + \frac{1}{2} \sum_{i=1}^N (y_{i+1} - y_i)(y_i - y_{i-1}) \\
&= 2 \int_0^1 r^*(x) d\lambda(x) + \frac{1}{2} \sum_{i=1}^N (y_{i+1} - y_i)(y_i - y_{i-1}) \\
&= 2SC + \frac{1}{2} \sum_{i=1}^N (y_{i+1} - y_i)(y_i - y_{i-1})
\end{aligned}$$

Define $\xi = \frac{1}{2} \sum_{i=1}^N (y_{i+1} - y_i)(y_i - y_{i-1})$. Notice that:

$$(y_{i+1} - y_i)(y_i - y_{i-1}) = 4 \left[y_i - \left(\frac{y_i + y_{i-1}}{2} \right) \right] \left[\left(\frac{y_i + y_{i+1}}{2} \right) - y_i \right]$$

that is, $(y_{i+1} - y_i)(y_i - y_{i-1}) = 4\pi_i^- \pi_i^+$, hence $(y_{i+1} - y_i)(y_i - y_{i-1}) = 4\gamma_i^2$. Therefore, $\xi = 2 \sum_{i=1}^N \gamma_i^2$, that is, $\xi = 2 \|\gamma\|^2$. Then $\mathcal{H}_N = 2SC + 2 \|\gamma\|^2$, which implies that $SC_{\min} = \frac{\mathcal{H}_N}{2} - \|\gamma\|^2$. We know that $\beta_i = \pi_i^- + \pi_i^+$, so that $\beta_i^2 = (\pi_i^-)^2 + (\pi_i^+)^2 + 2\pi_i^- \pi_i^+$. Summing over the firms, this implies that $\mathcal{H}_N = \sigma^- + \sigma^+ + 2 \|\gamma\|^2$. Substituting this into $SC_{\min} = \frac{\mathcal{H}_N}{2} - \|\gamma\|^2$, we get $SC_{\min} = \frac{\sigma^- + \sigma^+}{2}$, which proves the theorem. ■

From $\frac{d\mu^*(x, y_i)}{d\lambda} = \mathbf{1}_{B_i}(x)$ we have that $\mu^*(dx, y_i) = \mathbf{1}_{B_i}(x) \lambda(dx)$. By the same reasoning applied to the uniform case, this means that the disutility from transportation equals the social valuation of assigning consumer x to firm y_i . Similarly, $\frac{d\mu^*(x, y_i)}{d\nu} = \frac{1}{\beta_i} \mu^*(x, y_i)$ implies $\mu^*(x, dy_i) = \mu^*(x, y_i)$, so firms have zero shadow-price.

3 Hirschmann-Herfindahl index as a proxy to total transportation cost

In this section we determine the minimum and maximum total cost of transportation and show that these bounds are given by specific fractions of the Hirschmann-Herfindahl index. Therefore, the lower the concentration of the industry, the better the Hirschmann-Herfindahl index is an accurate proxy of the total cost.

Theorem 3: *Let $\mathbf{H}_N = (\lambda, \nu, c, N)$ be a Hotelling industry with market shares β_1, \dots, β_N and let $\mathcal{H}_N = \sum_{i=1}^N \beta_i^2$ be the Hirschmann-Herfindahl index. Let TC be the total cost of transportation. Then:*

- (a) $\frac{\mathcal{H}_N}{4} \leq SC \leq \frac{\mathcal{H}_N}{2}$
- (b) *the minimum social cost is achieved under uniform distribution of firms, that is, $\beta_i = \frac{1}{N}$, $\forall i = 1, \dots, N$, in which case $SC = \frac{1}{4N}$.*

Proof: (a) From theorem 2, $SC = \frac{\mathcal{H}_N}{2} - \|\gamma\|^2$. Then it follows that $SC \leq \frac{\mathcal{H}_N}{2}$. We know that $\beta_i = \pi_i^- + \pi_i^+$, hence $\frac{\beta_i}{2} = \frac{\pi_i^- + \pi_i^+}{2}$. Recall that $\sqrt{\pi_i^- \times \pi_i^+} \leq \frac{\pi_i^- + \pi_i^+}{2}$. Then $\gamma_i^2 \leq \frac{\beta_i^2}{4}$. Adding these terms up, we get $\sum_{i=1}^N \gamma_i^2 \leq \frac{1}{4} \sum_{i=1}^N \beta_i^2$, that is, $\|\gamma\|^2 \leq \frac{\mathcal{H}_N}{4}$. Therefore, $SC \geq \frac{\mathcal{H}_N}{2} - \frac{\mathcal{H}_N}{4}$, that is, $TC \geq \frac{\mathcal{H}_N}{4}$. Combining both inequalities, $\frac{\mathcal{H}_N}{4} \leq SC \leq \frac{\mathcal{H}_N}{2}$, which proves part (a).

(b) Since $SC = \frac{\sigma^- + \sigma^+}{2}$, we have that $SC = \frac{1}{2} \sum_{i=1}^N (\pi_i^-)^2 + \frac{1}{2} \sum_{i=1}^N (\pi_i^+)^2$. Therefore, the problem of minimizing the social cost SC consists in determining the vectors $\pi^- = (\pi_1^-, \dots, \pi_N^-)$ and $\pi^+ = (\pi_1^+, \dots, \pi_N^+)$ that solve the following constrained minimization:

$$\begin{cases} \min & \frac{1}{2} \sum_{i=1}^N (\pi_i^-)^2 + \frac{1}{2} \sum_{i=1}^N (\pi_i^+)^2 \\ \text{s.t.} & \sum_{i=1}^N \pi_i^- + \sum_{i=1}^N \pi_i^+ = 1 \end{cases}$$

Let $\Xi = \frac{1}{2} \sum_{i=1}^N (\pi_i^-)^2 + \frac{1}{2} \sum_{i=1}^N (\pi_i^+)^2 - \xi \left(\sum_{i=1}^N \pi_i^- + \sum_{i=1}^N \pi_i^+ - 1 \right)$ be the Lagrangean function. It is straightforward to see that the first-order condition, $\frac{\partial \Xi}{\partial \pi_i^-} = \frac{\partial \Xi}{\partial \pi_i^+} = 0$, implies $\pi_i^- = \pi_i^+ = \xi$. By the constraint, we get $\xi = \frac{1}{2N}$, so the solution is $\pi_i^- = \pi_i^+ = \frac{1}{2N}$. Since the constraint is linear and the objective-function is strictly convex, this is indeed a solution and is unique. Since $\beta_i = \pi_i^- + \pi_i^+$, the market shares that yield minimum total cost are given by $\beta_i = \frac{1}{N}$, $\forall i = 1, \dots, N$, that is, the uniform distribution of firms. Indeed, when the distribution of market shares in an industry with N firms is uniform, the Hirschmann-Herfindahl index is $\mathcal{H}_N = \frac{1}{N}$. In this case, SC achieves the minimum value, $\frac{1}{4N}$, as shown in the theorem 1. This completes the proof. ■

The corollary above shows that the Hirschmann-Herfindahl index of concentration is a proxy for the total transportation cost. The less concentrated an industry, the lower

its total cost and thus the higher its welfare. Besides, this proxy is more accurate when concentration is low. Of course the appearance of the Hirschmann-Herfindahl index is due to the particular cost function we considered. This brings to light the idea that total social costs can always be expressed in terms of some concentration measure derived by specific convex cost functions.

Let us make some remarks about the total cost. In a homogeneously distributed industry, $1 - y_1 = y_N = \frac{2N-1}{2N}$. Since $\alpha(N) = \sigma^-/y_N^2$, $\kappa(N) = \sigma^-(1 - y_1)^2$ and $SC_N = \frac{\sigma^- + \sigma^+}{2}$, we have that $SC_N = \frac{\alpha(N) + \kappa(N)}{2} \left(1 - \frac{1}{2N}\right)^2$, where we indexed the total cost by the size of the industry. Therefore, if the size of the industry is large, the total cost is approximately the average between anakinesis and katakinesis: $\lim_{N \rightarrow \infty} SC_N = \frac{\alpha(\infty) + \kappa(\infty)}{2}$, where $\lim_{N \rightarrow \infty} \alpha(N) = \alpha(\infty)$, similarly for $\kappa(\infty)$. On the other hand, $SC_N = \frac{1}{4N}$, then $\alpha(\infty) + \kappa(\infty) = 0$. Since $\alpha(\infty), \kappa(\infty) \geq 0$, we have $\alpha(\infty) = \kappa(\infty) = 0$. Of course this result is not exclusive to homogeneity, it only requires that $1 - y_1 = y_N \approx 0$.

We conjecture that the size of the industry also plays a role, though it is not so explicitly obvious. If two industries have the same degree of concentration, then the one with greater number of firms has lower total cost. This intuition is formalized as follows. Let $\mathbf{H}_N = (\lambda, \nu^N, c^N, N)$ be a Hotelling industry with size N , where we indexed the distribution of firms and the cost function by the size of the industry. Let $Y^{(N)} = \{y_1^N, \dots, y_N^N\}$ be the set of firms and assume that $0 < y_1^N < \dots < y_N^N < 1$. A *displacement* of $Y^{(N)}$ is any set $Z^{(N+1)} = \{\hat{y}_1^{N+1}, \dots, \hat{y}_{N+1}^{N+1}\}$ such that, $\forall k = 1, \dots, N+1$, $\hat{y}_k^N \in (y_{k-1}^N, y_k^N)$. A *dyadic refinement* of \mathbf{H}_N is a Hotelling industry $\rho(\mathbf{H}_N) = (\lambda, \nu^{2N+1}, c^{2N+1}, 2N+1)$ for which the set of firms is $Y^{(2N+1)} = Z^{(N+1)} \cup Y^{(N)}$ and the cost function $c(x, y) = \|x - y\|$ is defined on $X \times Y^{(2N+1)}$. Then we can write $Y^{(2N+1)} = \{y_1^{2N+1}, \dots, y_{2N+1}^{2N+1}\}$, where $y_k^{2N+1} = \hat{y}_{\frac{k+1}{2}}^N$, if k is odd, and $y_k^{2N+1} = y_{\frac{k}{2}}^N$, if k is even.⁷ In addition, $y_o^{2N+1} = 0$ and $y_{2N+2}^{2N+1} = 1$. Therefore, a dyadic refinement is a replica of the original industry to which intermediate firms are added, plus another one with higher characteristic than any other original firm. Not only every original firm now faces a new pair of neighbors, they see them even closer than the older ones. If we proceed this indefinitely, we get composite Hotelling industries given by $\rho^k(\mathbf{H}_N)$. Call it a *dyadic refinement of order k* . Of course $\rho^0(\mathbf{H}_N) = \mathbf{H}_N$. We then conjecture the following:

Conjecture: Let \mathbf{H}_N be a Hotelling industry and let $\rho^k(\mathbf{H}_N)$ be its dyadic refinement of order k . Suppose that all the $\rho^k(\mathbf{H}_N)$'s have the same Hirschmann-Herfindahl index H . Let $\tau^k = SC(\rho^k(\mathbf{H}_N))$ be the social costs of $\rho^k(\mathbf{H}_N)$. Then the sequence $\{\tau^k\}_{k=0}^\infty$ is strictly decreasing, that is, $\tau^k \downarrow 0$.

⁷Notice that $y_N^N < \hat{y}_{N+1}^N < 1$, so this explains why the dyadic refinement has $2N+1$ firms, instead of $2N$.

We know that $SC(\rho(\mathbf{H}_N)) = \frac{\mathcal{H}}{2} - \|\gamma(\nu^{2N+1}, 2N+1)\|^2$ and $SC(\mathbf{H}_N) = \frac{\mathcal{H}}{2} - \|\gamma(\nu^N, N)\|^2$, where $\gamma(\delta, L)$ is the vector γ corresponding to an industry with distribution δ and size L . Here, $\delta = \nu^N, \nu^{2N+1}$ and $L = N, 2N+1$. Define $\Delta_N = SC(\mathbf{H}_N) - SC(\rho(\mathbf{H}_N))$. Then $\Delta_N = \|\gamma(\nu^{2N+1}, 2N+1)\|^2 - \|\gamma(\nu^N, N)\|^2$. Recall that $\|\gamma(\nu^N, N)\|^2 = \frac{1}{4} \sum_{i=1}^N (y_{i+1}^N - y_i^N)(y_i^N - y_{i-1}^N)$ and, similarly, $\|\gamma(\nu^{2N+1}, 2N+1)\|^2 = \frac{1}{4} \sum_{i=1}^{2N+1} (y_{i+1}^{2N+1} - y_i^{2N+1})(y_i^{2N+1} - y_{i-1}^{2N+1})$. Therefore, $4\Delta_N = \sum_{i=1}^{2N+1} (y_{i+1}^{2N+1} - y_i^{2N+1})(y_i^{2N+1} - y_{i-1}^{2N+1}) - \sum_{i=1}^N (y_{i+1}^N - y_i^N)(y_i^N - y_{i-1}^N)$. Thus the conjecture would be proved true if we show that $4\Delta_N > 0$, since, by the assumption of constant Hirschmann-Herfindahl index, the inequality would hold for every step. Proving this, however, is out of the scope of this paper.

4 Variations

In this section we present some variations from the Hotelling linear city model and show how they can be reframed as Monge-Kantorovich problems. We analyze the case of multidimensional characteristics, vertical product differentiation, Bertrand games, and non-uniform distributions of consumers. For the particular case of the linear city model, we also show how the optimal social cost can be interpreted as a metric between the distribution of firms and of consumers.

4.1 Optimal social cost and 1-Wasserstein metric

There is a strong link between the total transportation cost for the Euclidean metric $\|x - y\|$ and the 1-Wasserstein metric on the space of measures. Consider the set $\Pi(\lambda, \nu) \subset \mathbf{M}_+(X \times Y)$ of feasible measures endowed with the relative weak topology. Since $Y \subset X$, we can regard ν as an atomic measure defined on $\mathcal{F} = \mathcal{B}[0, 1]$. Thus we can assume that $\lambda, \nu \in BV[0, 1]$. The 1-Wasserstein metric on $BV[0, 1]$ is defined by:

$$W_1(\lambda, \nu) = \inf \left\{ \int_X \int_Y \|x - y\| d\mu(x, y) : \mu \in \Pi(\lambda, \nu) \right\}$$

Therefore, the total cost of transportation can be interpreted as a distance between the distribution of consumers and the distribution of firms. In other words, welfare in the Hotelling industry increases whenever consumers find it easier (in distance terms) to move to the nearest firm. But this is not enough. Welfare will increase even more if such distributional similarity approaches the uniform distribution. Therefore, two factors make the Hotelling industry less efficient: industry concentration and the transportation cost. We have shown that:

$$W_1(\lambda, \nu) = \frac{\mathcal{H}_N}{2} - \|\gamma\|^2$$

In other words, given the distribution of firms and given the distance between the distribution of consumers and the distribution of firms, according to the 1-Wasserstein metric, we have that $\|\gamma\|^2 = \frac{\mathcal{H}_N}{2} - W_1(\lambda, \nu)$, which gives us at least a possible interpretation for the extra term $\|\gamma\|$. We know that \mathcal{H}_N is a measure of the concentration of sellers. The greater the index \mathcal{H}_N , the more the sellers are positioned close to each other in a distributional sense. If the distribution of consumers is too far from the distribution of firms, that is, if $W_1(\lambda, \nu)$ is big, then the smaller the value $\|\gamma\|^2$, hence the smaller the amount subtracted from the upper bound for the social cost. So it measures the degree of difficulty consumers have to move themselves to the firms they are assigned to. However, this degree of difficulty is reduced if firms are positioned in a more uniform fashion.

4.2 Multidimensional characteristics

Our framework, though mathematically heavier than the usual ones existing in the literature, is simple enough to allow for many modifications, provided they are modeled linearly. For example, the bidimensional model also fits our framework, provided we redefine the underlying spaces and the cost function. In this model we have consumers and firms located on a planar city. So consumers like two characteristics. For instance, a depositor may choose a certain bank because of the package of services it offers. In this case, $X = [0, 1] \times [0, 1]$ and $Y = \{y_1, \dots, y_N\} \subset [0, 1] \times [0, 1]$, where $y_i \in [0, 1] \times [0, 1]$. The cost function may be any convex function $c((x_1, x_2), y)$ defined on $[0, 1] \times [0, 1] \times Y$. Then the problem is:

$$\left\{ \begin{array}{l} \min \quad \sum_{i=1}^n \int_0^1 \int_0^1 c((x_1, x_2), y_i) d\mu((x_1, x_2), y_i) \\ s.t. \quad \mathbb{P}_{[0,1] \times [0,1]}[\mu] = \lambda_2 \\ \quad \mathbb{P}_Y[\mu] = \nu \\ \quad \mu \in \mathbf{M}_+([0, 1] \times [0, 1] \times Y) \end{array} \right.$$

where λ_2 is the Lebesgue measure on $[0, 1] \times [0, 1]$ and $\nu(y_i) = \beta_i$ is firm i 's market share. This is a semi-infinite transportation problem and can be resolved by means of its dual problem just like we did. If s_i is firm i 's shadow-price and $r(x_1, x_2)$ is the shadow-price of the consumer (x_1, x_2) , then the dual problem is:

$$\left\{ \begin{array}{l} \max \quad \sum_{i=1}^n \beta_i s_i + \int_0^1 \int_0^1 r(x_1, x_2) d\lambda_2(x_1, x_2) \\ s.t. \quad r(x_1, x_2) + s_i \leq c((x_1, x_2), y_i) \\ \quad \forall (x_1, x_2) \in [0, 1] \times [0, 1] \text{ and } \forall i = 1, \dots, n \\ \quad r \in \mathbf{C}([0, 1] \times [0, 1]) \end{array} \right.$$

4.3 Vertical product differentiation

We can also reframe vertical differentiation by assuming that the set of consumers, $X \subset \mathbb{R}^m$, is a compact separable metric space and that the space of characteristics is given by a finite set $Y \subset \mathbb{R}^m$ for which $Y - X \subset \mathbb{R}_+^m$, the positive cone of the m -dimensional Euclidean space. Indeed, under vertical product differentiation, there is a unanimous quality ranking [Martin (1993), p. 261]. In other words, higher characteristics on their scale of measurement are regarded as better by all consumers. This role is played by the partial order defined by the condition $Y - X \subset \mathbb{R}_+^m$, which means that, $\forall x \in X$ and $\forall y \in Y$, $y \geq x$, where \geq is the componentwise inequality.

4.4 A rôle for firms

We gave no technological role to the firms, since our interest focused only on the issue of transportation. However, this is not an oversimplification, it would only add some extra function, say $\varphi(x, y)$, to the cost function $\|x - y\|$. For instance, we could assume that $\varphi(x, y) = kx$ is the cost function of firm y , where k is a constant marginal cost. If firm y sells to a coalition B of consumers, with Lebesgue measure $\lambda(B) > 0$, then its total cost is $k\lambda(B)$. However, the effect of this will be just a modification of the right-hand side of the dual constraints. We would get then $\varphi(x, y) + \|x - y\|$, but this would make no difference to our main contribution. In this case, the primal problem is:

$$\left\{ \begin{array}{l} \inf \quad \int_X \int_Y (\varphi(x, y) + \|x - y\|) d\mu(x, y) \\ s.t. \quad \mathbb{P}_X[\mu] = \lambda \\ \quad \mathbb{P}_Y[\mu] = \nu \\ \quad \mu \in \mathbf{M}_+(X \times Y) \end{array} \right.$$

and the corresponding dual:

$$\left\{ \begin{array}{l} \sup \quad \int_X r(x) d\lambda(x) + \int_Y s(y) d\nu(y) \\ s.t. \quad r(x) + s(y) \leq \varphi(x, y) + \|x - y\|, \forall (x, y) \in X \times Y \\ \quad r \in \mathbf{C}(X), s \in \mathbf{C}(Y) \end{array} \right.$$

Therefore, there is no significant change in the overall formalization of the problem.

4.5 Bertrand games

If we had considered a Bertrand game, then we could assume that consumer $x \in (y_{i-1}, y_i)$ have utility function $U(x, y_{i-1}) = b(x) - p_{i-1} - (x - y_{i-1})$, for $x \in (y_{i-1}, \frac{y_i + y_{i-1}}{2}]$, and $U(x, y_i) = b(x) - p_i - (y_i - x)$, for $x \in [\frac{y_i + y_{i-1}}{2}, y_i)$, where p_{i-1} is the price charged

by firm y_{i-1} , p_i is the price charged by firm y_i , and $b(x)$ is the benefit, to consumer x , from the consumption of one unit of the good. In this case, the utility function is $U(x, y_i) = \sum_{i=0}^N (b(x) - p_i - \|x - y_i\|) \mathbf{1}_{B_i}(x)$. Notice that the primal objective-function would be the aggregate surplus, $\int_X \int_Y (p_i \mathbf{1}_{B_i}(x) + (b(x) - p_i) \mathbf{1}_{B_i}(x) - \|x - y\|) d\mu(x, y)$, which is equivalent to $\int_X \int_Y (b(x) \mathbf{1}_{B_i}(x) - \|x - y\|) d\mu(x, y)$. Since $\{B_i\}_{i=1}^N$ is a partition of $[0, 1]$ and consumers in B_i move to $y \in B_i$, this is equal to $\int_X \int_Y (b(x) - \|x - y\|) d\mu(x, y)$, that is, the whole problem would be one of maximization of consumers' surplus. If we assume that $b(x) = b$, $\forall x \in [0, 1]$, and that the equilibrium is symmetric, $p_{i-1} = p_i$, then the net benefit (transportation costs aside) is constant and plays no role in the allocational problem. Otherwise we would just have to add the extra term $(b(x) - p_i) \mathbf{1}_{B_i}(x)$ to the righthand side of the dual constraint, that is, $r(x) + s(y) \geq (b(x) - p_i) \mathbf{1}_{B_i}(x) + \|x - y\|$, $\forall (x, y) \in X \times Y$, and find the optimal distributional Hotelling allocation.⁸ The inequality was reversed because the dual problem is now a minimization. Indeed, the primal problem would be:

$$\begin{cases} \max & \int_X \int_Y (b(x) - \|x - y\|) d\mu(x, y) \\ s.t. & \mathbb{P}_X[\mu] = \lambda \\ & \mathbb{P}_Y[\mu] = \nu \\ & \mu \in \mathbf{M}_+(X \times Y) \end{cases}$$

and the corresponding dual:

$$\begin{cases} \inf & \int_X r(x) d\lambda(x) + \int_Y s(y) d\nu(y) \\ s.t. & r(x) + s(y) \geq b(x) - \|x - y\|, \forall (x, y) \in X \times Y \\ & r \in \mathbf{C}(X), s \in \mathbf{C}(Y) \end{cases}$$

We would afterwards have to calculate the profit of each firm, its payoff, and to determine the corresponding Nash equilibrium. This would only complicate matters without changing anything in the Kantorovich framework inherent to the Hotelling's model. Indeed, the Kantorovich framework is solely related to the assignment of consumers to firms, and has nothing to do with the strategic choice of prices in a Bertrand competition. Once the Hotelling allocation is in place, profits can be calculated and only then a Nash equilibrium can be found. This reasoning applies to the case of concentrated industries as well, so we will not return to this issue again in this paper.

4.6 Non-uniform distribution of consumers

We can also change the distribution of consumers, which is just another way to saying that the distribution of tastes is non-uniform. Suppose, for instance, that the distribution

⁸The term $\mathbf{1}_{B_i}(x)$ can actually be neglected, since the partition will be found in the solution process anyway.

of consumers' tastes is non-uniform, say, a measure ρ which is absolutely continuous with respect to the Lebesgue measure, that is, $\rho \ll \lambda$, and whose density function (or Radon-Nikodym derivative with respect to λ) is given by $f(x) = 2x$, for $x \in [0, 1]$. Since $d\rho(x) = 2x d\lambda(x)$, it is easy to see that the dual problem for the heterogenous case becomes:

$$\begin{cases} \max & \int_0^1 r(x) d\rho(x) + \sum_{i=1}^N \beta_i s_i \\ \text{s.t.} & r(x) + s_i \leq \|x - y_i\|, \forall x \in [0, 1], \forall i = 1, \dots, N \\ & r \in \mathbf{C}[0, 1] \end{cases}$$

where $r(x)$ is the shadow-price of consumer x and s_i is the shadow-price of firm i . It is easy to show⁹ that firms get zero shadow-prices and consumers get shadow-prices according to the function $r^*(x) = \sum_{i=1}^N 2x \|x - y_i\| \mathbf{1}_{B_1}(x)$.

5 Conclusion

We provided a novel framework to study the Hotelling's linear city model. We showed that it can be written as a special type of the Monge-Kantorovich mass transportation problem, which, on its turn, is an infinite-dimensional linear programming problem. By means of the dual optimization associated with the primal problem of minimum total transportation cost, we solved the Hotelling's problem in a different way and, as a by-product, we got the shadow prices of consumers and firms. In addition, our solution can also be expressed in terms of its Radon-Nikodym derivatives, which are intimately related with the determination of shadow-prices. From the Radon-Nykodym partial derivatives of the optimal distributional allocation with respect to the distribution of consumers and firms, it is possible to find out the shadow-prices of consumers and firms, respectively.

The contribution of the reformulation is double. First, under the cost of treating the Hotelling model in a more difficult mathematical way, we get the benefit of providing

⁹The problem is equivalent to making $s_i = \inf\{2x \|x - y_i\| - r(x) : x \in [0, 1]\}$. Define $\psi_i(x) = 2x \|x - y_i\| - r(x)$. If $x > y_i$, then, in a sufficiently small neighborhood of y_i , we have $\psi_i(x) = 2x(x - y_i) - r(x)$. In this case, $\frac{\partial \psi_i}{\partial x} = 0$ implies $4x - 2y_i - r'(x) = 0$, so that $r'(x) = 4x - 2y_i$, hence $r(x) = 2x^2 - 2y_i x + A_i$, where A_i is a constant of integration. If $x < y_i$, then $\psi_i(x) = 2x(-x + y_i) - r(x)$. Thus, $\frac{\partial \psi_i}{\partial x} = 0$ implies $-4x + 2y_i - r'(x) = 0$, so that $r'(x) = -4x + 2y_i$, that is, $r(x) = -2x^2 + 2y_i x + C_i$. Since r is continuous, we have $r(y_i+) = r(y_i-)$, that is, $\lim_{x \uparrow y_i} r(x) = \lim_{x \downarrow y_i} r(x)$. Therefore, $2y_i^2 - 2y_i^2 + A_i = -2y_i^2 + 2y_i^2 + C_i$. Then $C_i = A_i$. Since $r(y_i) = 0$, we have $C_i = A_i = 0$, for all $i = 1, \dots, n$. The individual indifferent between two adjacent locations y_i and y_{i+1} must satisfy the equation $2x \|x - y_{i+1}\| = 2x \|x - y_i\|$. In this case, she is the one halfway between the firms. The solution is: $r(x) = 2x \|x - y_i\|$, for $x \in [\frac{y_{i-1}+y_i}{2}, \frac{y_i+y_{i+1}}{2})$. In other words, $r^*(x) = \sum_{i=1}^N 2x \|x - y_i\| \mathbf{1}_{B_2}(x)$. This still implies that $s_i^* = 0$. In addition, $\sum_{i=1}^N \int_0^1 \|x - y_i\| d\mu^*(x, y_i) = \sum_{i=1}^N \int_0^1 2x \|x - y_i\| \mathbf{1}_{B_i}(x) d\lambda(x)$. The optimal Hotelling-allocation is then characterized, in terms of one of its Radon-Nikodym derivatives, by $d\mu^*(x, y_i) = 2x \mathbf{1}_{B_i}(x) d\lambda(x)$.

a fairly general framework which comprehends many variations of the Hotelling model. Second, this new framework, besides corroborating results given by standard approaches, highlights the usefulness of Radon-Nikodym derivatives and duality theory to reinterpret the optimal allocation in terms of shadow-prices.

We hope to have made a point in favor of the use of infinite-dimensional linear programming in the Hotelling model of product differentiation. Its main advantage is the determination of shadow-prices of agents and its being a general framework capable of encompassing a variety of modifications to the standard linear city model, such as horizontal and vertical product differentiation and high dimensional versions.

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